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GROWTH OF TRANSCENDENTAL ENTIRE FUNCTIONS ON ALGEBRAIC VARIETIES

BY

STANLEY M. EINSTEIN-MATTHEWS AND CLEMENT H. LUTTERODT*

Department of mathematics, Howard University
2441 6th. Street, N.W., Washington, DC 20059, USA
e-mail: seinstein-matth@Howard.edu clutterodt@Howard.edu

ABSTRACT

Let X be a complete intersection algebraic variety of codimension $m > 1$ in $\mathbb{C}^{m+n}$. We define the notion of (p,q)-order and (p,q)-K-type for transcendental entire functions $f \in \mathcal{O}(\mathbb{C}^{m+n})$ where K is a non-pluripolar compact subset of $\mathbb{C}^{m+n}$. Further, we consider the analogues of (p,q)-order and (p,q)-K-type in $\mathcal{O}(X)$. We discuss the series expansions of the functions in $\mathcal{O}(X)$ in terms of an orthogonal basis in a Hilbert space $\mathcal{L}^2(X,\mu)$, where μ is a capacitary extremal measure on K.

1. Introduction

The present work was inspired by the seminal paper of Zeriahi [Ze.1]. The key ideas of order and type and the role these play in the study of the growth of transcendental entire functions in $\mathbb{C}^{m+n}$ are classical in complex analysis. The one complex variable case is well represented in the work of B. Ja. Levin [Lev.1]. In several complex variables the standard reference is now the work of P. Lelong and L. Gruman [Le-Gr.1] and, with a slightly different emphasis, Ronkin's book [Ro.1].

Our goal in this paper is to extend the notions of (p,q)-order and (p,q)-type studied in [E-M, K.1] to transcendental entire functions $f\colon X \to \mathbb{C}$, defined on a complete intersection algebraic variety X in $\mathbb{C}^{m+n}$ of codimension $m > 1$. We also introduce the new concept of (p,q)-K-type in our characterization of the growth properties of these transcendental entire functions. The subset K we use in the definition of (p,q)-K-type is assumed to be compact, nonpluripolar

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and a $\mathcal{L}$-regular subset of X. $\mathcal{L} = \mathcal{L}(\mathbb{C}^{m+n})$ and $\mathcal{L} = \mathcal{L}(X)$ are the Siciak and Sadullaev subfamilies of the plurisubharmonic (in short, psh) cones denoted by $\mathcal{PSH}(\mathbb{C}^{m+n})$ and $\mathcal{PSH}(X)$, respectively. The definitions of these cones, extremal functions associated to compact nonpluripolar sets and their important properties are given in Section 2. Section 3 is devoted to the proofs of Zeriahi-type theorems. In the final section of the paper we introduce the generalizations of the concepts of (p,q)-order and (p,q)-K-type to varieties, and state and prove the main result of this paper on the characterization of the growth properties of transcendental entire functions on complete intersection algebraic varieties.

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2. Preliminaries on (p,q)-order and (p,q)-K-type

In this section we recall some central classical notions of order and type, (p,q)-order and (p,q)-type, where p, and q are non-negative integers such that $p \geq q \geq 1$. Most of these results have been discussed in greater detail in the paper, [E-M, K.1], but for ease of reference we state some of the definitions and key lemmata here. As in, [E-M, K.1], let δ: $\mathbb{C}^{m+n} \to \mathbb{R}_+ := \{r \in \mathbb{R}\colon r > 0\}$ be a real-valued function such that the following properties hold:

(i) $\delta(z+w) \leq \delta(z) + \delta(w), \forall z, w \in \mathbb{C}^{m+n}$,

(ii) $\delta(bz) = |b|\delta(z), \forall z \in \mathbb{C}^{m+n}, \forall b \in \mathbb{C}$,

(iii) $\delta(z) = 0 \iff z = 0$. In this case δ is a norm on $\mathbb{C}^{m+n}$ and it exhausts the complex space $\mathbb{C}^{m+n}$ by a family of sublevel sets $\{\Omega_c\}_{c\geq 1}$ which are defined by

$$\Omega_c := \{z \in \mathbb{C}^{m+n}\colon \delta(z) \leq c, c \in \mathbb{R}\}.$$

Let Ψ: $\mathbb{C}^{m+n} \to \mathbb{R}_+$. Define the maximum of Ψ with respect to the norm δ by $\mathcal{M}_{\Psi,\delta}(r) := \sup_{\delta(z)\leq r} \Psi(z)$, for each $r \in \mathbb{R}_+$. Let f: $\mathbb{C}^{m+n} \to \mathbb{C}$ be a transcendental entire function on $\mathbb{C}^{m+n}$. Then, we say that f is of order ρ, if $\log|f|$ is of order ρ, where

$$\rho := \limsup_{r\to\infty} \frac{\log(\mathcal{M}_{f,\delta}(r))}{\log r}. \tag{2.1}$$

If $\rho < +\infty$, f is said to have maximal, normal or minimal type if

$$\sigma := \limsup_{r\to\infty} \frac{\mathcal{M}_{f,\delta}(r)}{r^\rho} \tag{2.2}$$

is infinite, finite or zero, and σ is said to be the type of the function f with respect to the norm δ.

Definition 2.1: A proximate order $\rho(r)$ for the order $\rho \geq 0$ is a function $\rho(r) \geq 0$ defined for each $r \in \mathbb{R}_+$ such that

(I) $\lim_{r\to\infty} \rho(r) = \rho$ and

(II) $\lim_{r\to\infty} \rho'(r) r \log r = 0$, where $\rho'(r)$ is the derivative of $\rho(r)$ with respect to r. If

$$\sigma = \limsup_{r\to\infty} \frac{\mathcal{M}_{f,\delta}(r)}{r^{\rho(r)}}$$

is finite, we say that $\rho(r)$ is the proximate order of the transcendental entire function $f\colon \mathbb{C}^{m+n} \to \mathbb{C}$

Definition 2.2: A transcendental entire function $f\colon \mathbb{C}^{m+n} \to \mathbb{C}$ is said to be of (p,q)-order ρ if

$$\rho \equiv \rho(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} \mathcal{M}_{f,\delta}(r)}{\log^{[q]} r}, \tag{2.3}$$

where $\log^{[s]} y = \exp^{[-s]} y = \log^{[s-1]}(\log y) = \exp(\exp^{[-s-1]} y), s = 0, \pm 1 \pm 2, \ldots,$ provided that $0 < \log^{[s-1]} < \infty$ and $\log^{[0]} y = \exp^{[0]} y = y$ with p, q integers such that $p \geq q \geq 1$.

Definition 2.3: A transcendental entire function $f\colon \mathbb{C}^{m+n} \to \mathbb{C}$ is of index-pair $(p,q), p \geq q \geq 1$, if $b < \rho(p,q) < \infty$ and $\rho(p-1, q-1)$ is not a nonzero finite number where $b = 1$ if $p = q$ and $b = 0$ if $p > q$.

(I) If $\rho(p,q)$ is never greater than 1 and $\rho(p', p') = 1$ for some integer $p' \geq 1$, then the index-pair of $f(z)$ is defined as (s,s) where

$$s := \inf\{p' \colon \rho(p', p') = 1\}.$$

(II) If $\rho(p,q)$ is never nonzero finite and $\rho(p'', 1) = 0$ for some integer $p'' \geq 1$, then the index-pair of $f(z)$ is defined as $(k, 1)$, where

$$k := \inf\{p'' \colon \rho(p'', 1) = 0\}.$$

(III) If $\rho(p,q)$ is always infinite, then the index-pair of $f(z)$ is defined to be (∞, ∞).

(IV) If $f(z)$ has the index-pair (p,q), then $\rho \equiv \rho(p,q)$ is called the (p,q)-order of $f(z)$.

To compare the growth of transcendental entire functions having the same (p,q)-order, we introduce the concept of (p,q)-type as follows:

Definition 2.4: A transcendental entire function f on $\mathbb{C}^{m+n}$ of (p,q)-order ρ $(b<\rho<\infty)$ is said to be of (p,q)-type σ if

$$\sigma \equiv \sigma(p,q) := \limsup_{r\to\infty} \frac{\log^{[p-1]} \mathcal{M}_{f,\delta}(r)}{\left(\log^{[q-1]} r\right)^{\rho}}, \quad 0 \le \sigma \le \infty, \tag{2.4}$$

where $b=1$ if $p=q$ and $b=0$ if $p>q$.

Definition 2.5: A transcendental entire function f, having index-pair (p,q), is said to be of (p,q)-growth $\{\rho,\sigma\}$ if it is of (p,q)-order that does not exceed ρ and its (p,q)-type does not exceed σ if it is of (p,q)-order ρ.

Definition 2.6: A positive function $\rho(r)$ defined on $[r_0,\infty], r_0 \ge \exp^{[q-1]}1$, is a proximate order for a transcendental entire function f on $\mathbb{C}^{m+n}$ with index-pair (p,q) if:

(I) $\rho(r) \to \rho(p,q) \equiv \rho$ as $r\to\infty, b<\rho<\infty$.

(II) $\bigwedge_{[q]}(r)\rho' \to 0$ as $r\to\infty, b=1$ if $p=q, b=0$ if $p>q$ and $\bigwedge_{[r]}(r) := \prod_{j=1}^{q} \log^{[j]} r$, where $\rho'(r)$ is the derivative of $\rho(r)$.

If, in addition to conditions (I) and (II), we have for $b<\rho<\infty$ that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} \mathcal{M}_{f,\delta}(r)}{\left(\log^{[q-1]} r\right)^{\rho(r)}} = \sigma(p,q) \equiv \sigma, \quad 0 \le \sigma \le \infty, \tag{2.5}$$

then $\rho(r)$ is said to be the proximate order of the trancendental entire function $f(z)$ if σ is nonzero and finite.

LEMMA 2.7 (E-M, K.1): *$(\log^{[q-1]} r)^{\rho(r)-A}$ is a monotone increasing function of r for $0<r_0<r<\infty$, where $A=1$ if $q=2$ and zero otherwise.*

Since $(\log^{[q-1]} r)^{\rho(r)-A}$ is a monotone increasing function of r, we can define a real-valued function $\psi(x)$ of $x>x_0>0$ to be a unique solution of the equation

$$x = \left(\log^{[q-1]} r\right)^{\rho(r)-A} \iff \psi(x) = \log^{[q-1]} r. \tag{2.6}$$

To introduce the important concept of (p,q)-K-type for trancendental functions on $\mathbb{C}^{m+n}$ we need some precise information on the $\mathbb{R}_+$-convex cone of plurisubharmonic functions (in short, psh functions) denoted by $PSH(.)$, and on $\mathbb{C}^{m+n}$ write $\mathcal{P}SH(\mathbb{C}^{m+n})$. In particular, we are interested in the subcones of extremal functions studied by Siciak [Si.1] on $\mathbb{C}^{m+n}$ and Sadullaev [Sd.1, 2] on the complete intersection variety X in $\mathbb{C}^{m+n}$. Let $\mathcal{L} := \mathcal{L}(\mathbb{C}^{m+n})$ be the class of all functions u which are plurisubharmonic on $\mathbb{C}^{m+n}$ and satisfy the condition

$$u(z) \leq \log(1 + \|z\|) + O(1), \quad \text{as } \|z\| \to +\infty, \tag{2.7}$$

where $\|.\|$ is the Euclidean norm on $\mathbb{C}^{m+n}$. This is the space of plurisubharmonic functions of logarithmic growth. Since $\mathcal{M}(r) := \sup_{\|z\|=R} u(z)$ is a convex, increasing function of $R \in \mathbb{R}$, we see easily that $\mathcal{L}$ consists of plurisubharmonic functions of minimal growth. For K a compact subset of $\mathbb{C}^{m+n}$, we define the Siciak extremal function associated to K by

$$V_K(\zeta) := \sup\{v(\zeta)\colon v \in \mathcal{L}; v(\zeta) \leq 0, \forall \zeta \in K\}. \tag{2.8}$$

Let

$$V_K^*(z) := \limsup_{\zeta \to z} V_K(\zeta) \tag{2.9}$$

be the upper semi-continuous regularization of V_K. This function has been studied extensively by Siciak [Si.1] and Sadullaev [Sd.1, 2]. The function V_K is in general not smooth on $\mathbb{C}^{m+n} \setminus K$ when $m+n > 0$. It is a theorem of Siciak [Si.1] that either $V_K^* \equiv +\infty$, in which case the set K is pluripolar, or else V_K^* is psh and

$$V_K^*(z) \leq \log(\|z\| + 1) + O(1), \quad \text{as } \|z\| \to +\infty. \tag{2.10}$$

If V_K is continuous on $\mathbb{C}^{m+n}$, then $V_K = V_K^* \in \mathcal{L}$. In the paper [Sd.2], Sadullaev studied the case of these extremal psh functions on a complete intersection algebraic variety, X of codimension $m > 1$ in $\mathbb{C}^{m+n}$. This variety is, in fact, the intersection of m hypersurfaces that are transverse at each point of their intersection. If K is a compact subset of $\mathbb{C}^{m+n}$ which is nonpluripolar on each irreducible component of a complete intersection variety X, then the upper semi-continuous regularization of V_K can be defined on X by

$$V_K^*(z) := \limsup_{\zeta \to z} V_K(\zeta), \quad \zeta \in K,\ z \in X. \tag{2.11}$$

This function is psh on X and satisfies

$$V_K^*(z) \leq \log(\|z\| + 1) + O(1), \tag{2.12}$$

where

$$V_K(z) := \sup\{v(z): v \in \mathcal{L}(X); v(\zeta) \leq 0, \forall \zeta \in K, z \in X\}. \tag{2.13}$$

The subcone $\mathcal{L}(X)$ is given by

$$\mathcal{L}(X) := \{v(z): v \in \mathcal{PSH}(X); v(z) \leq \log(\|z| + 1) + C_v, z \in X\}, \tag{2.14}$$

where C_v is a constant depending only on the psh function v. It is a result of Sadullaev that if, for all $z \in K, V_K$ is continuous, then V_K is continuous on X. In this case we say that K is $\mathcal{L}$-regular in X. We define the sublevel sets of the extremal function V_K by setting

$$\Omega_\alpha := \{z \in X: V_K(z) \leq \alpha\}, \quad \alpha > 1,\ \alpha \in \mathbb{R}. \tag{2.15}$$

Suppose K is a compact nonpluripolar subset of $\mathbb{C}^{m+n}$, and that the associated function V_K to K is $\mathcal{L}$-extremal on $\mathbb{C}^{m+n}$; we can define the sublevel sets of the upper semi-continuous regularization V_K^* of V_K by

$$\Omega_r := \{z \in \mathbb{C}^{m+n}: \exp V_K^*(z) < r\}, \quad r > 1. \tag{2.16}$$

Observe that the sequence of sublevel sets $\{\Omega_r\}_{r>1}$ exhausts the complex space $\mathbb{C}^{m+n}$. For $f: \mathbb{C}^{m+n} \to \mathbb{C}$ a transcendental entire function, set

$$\mathcal{M}_{K,f}(r) := \sup_{z \in \overline{\Omega}_r} |f(z)|, \quad r > 1. \tag{2.17}$$

One shows easily that $\log^+ \mathcal{M}_{K,f}$ and $\log^+ \mathcal{M}_{f,\delta}(r)$ give the same order, called the order of the transcendental entire function f on $\mathbb{C}^{m+n}$ and given by

$$\rho \equiv \rho(f) := \limsup_{r \to \infty} \frac{\log \log^+ \mathcal{M}_{K,f}(r)}{\log r}. \tag{2.18}$$

Most of the definitions that follow are inspired by, and in fact in some sense are contained in, Zeriahi's paper [Ze.1]. This paper had its origin in trying to understand the cases studied by Zeriahi. If the order of f, $\rho = \rho(f)$, is nonzero and finite ($0 < \rho < \infty$), we define the K-type of the transcendental entire function f on $\mathbb{C}^{m+n}$ by

$$\sigma_K \equiv \sigma_K(f) \equiv \sigma_{K,f} := \limsup_{r \to \infty} \frac{\log^+ \mathcal{M}_{K,f}(r)}{r^\rho}. \tag{2.19}$$

The definitions of order and K-type given in (2.18) and (2.19) easily generalize to the case of transcendental entire functions on the complete intersection algebraic variety X of codimension $m > 1$ in $\mathbb{C}^{m+n}$. The definition for the case of proximate order $\rho(r)$ similarly generalizes to this case. Let K be a compact, nonpluripolar and $\mathcal{L}$-regular subset of X, and let μ be the extremal capacitary measure on K given by

$$\mu := \left(dd^c V_K^*\right)^n. \tag{2.20}$$

It is known that for a nonpluripolar set K this measure is a positive Borel measure supported on K. Further details on this can be obtained from the interesting paper of E. Bedford and B. A. Taylor [B-T.1].

Here we reformulate the definitions of (p,q)-order and (p,q)-K-type in a form most suitable for our purposes. We shall need this formulation in the final section of the paper. Let $f\colon X \to \mathbb{C}$ be a transcendental entire function on a complete intersection variety X. Then we write the (p,q)-order ρ and the (p,q)-K-type in the forms:

$$\rho \equiv \rho(p,q) := \limsup_{r\to\infty} \frac{\log^{[p]} \mathcal{M}_{K,f}(r)}{\log^{[q]} r} \tag{2.21}$$

and

$$\begin{aligned}\sigma \equiv \sigma_K \equiv \sigma_K(p,q)&\\ := \limsup_{r\to\infty} \frac{\log^{[p-1]} \mathcal{M}_{K,f}(r)}{\left(\log^{[q-1]} r\right)^{\rho(r)}},&\quad 0 \le \sigma \le \infty,\ p \ge q \ge 1.\end{aligned} \tag{2.22}$$

3. Proofs of Zeriahi-type theorems

This section provides new proofs of the Zeriahi-type theorems. In his paper, Zeriahi refers to the book by Boas [Bo.1] for the rest of the proof to his theorems. Since this book deals with the one-variable case and is often not a current reference source in most libraries, we felt the need to provide here, in compact form, independent proofs of these theorems. In any case, we required a knowledge of the structure of the proofs as an aid to the generalizations to the case of (p,q)-order and (p,q)-K-type on complete intersection algebraic varieties in $\mathbb{C}^{m+n}$. In the paper [Ze.1], Zeriahi constructed an orthogonal polynomial basis $\{A_k\}_{k\ge 1}$ for the space $\mathcal{O}(X)$ of holomorphic functions on the complete intersection algebraic variety X. The basis is orthogonal in the Hilbert space $L^2(X,\mu)$, essentially by means of the Hilbert–Schmidt process. This basis is carefully constructed from the sequence $\{e_k\}_{k\geq 1}$ of linearly independent polynomial monomials on X defined by

$$e_j(z) := z^{\alpha(j)}, \quad z \in X,\ j \in \mathbb{N}. \tag{3.1}$$

The details of this construction can be found in Section 3 of Zeriahi's article [Ze.1].

We can now state Zeriahi's Bernstein–Markov type Inequality:

BM: $\forall \epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$\sup_{z\in K} |f(z)| \leq C_\epsilon (1+\epsilon)^{\deg(f)} \Big(\int_K |f|^2 d\mu \Big)^{1/2} \tag{3.2}$$

for every holomorphic function f with polynomial growth on the complete intersection algebraic variety X and K is a nonpluripolar compact subset of X.

Set

$$\Delta_k(K) = \Big(\int_K |A_k|^2 d\mu \Big)^{1/2}, \quad k \geq 1 \tag{3.3}$$

and

$$a_k(K) := \max_{z\in K} |A_k(z)|, \quad k \geq 1. \tag{3.4}$$

If the extremal function V_K associated with K is continuous for every $z \in K$, then V_K is continuous on X and $\mathcal{L}$-regular, so instead of defining sublevel sets for the upper semi-continuous regularization we might just as well define the same for V_K by setting

$$\Omega_r := \{z \in X\colon V_K(z) < \log r, r \in \mathbb{R}, r > 1\}. \tag{3.5}$$

Then we have

$$V_K(z) \geq \frac{1}{s_k} \log \Big(\frac{|A_k|}{a_k(K)} \Big), \tag{3.6}$$

where

$$|A_k|_{\overline{\Omega}_r} \leq a_k(K) r^{s_k}, \quad s_k := \text{degree}(A_k). \tag{3.7}$$

Let $\mathcal{P}_d(\mathbb{C}^{m+n})$ denote the the $\mathbb{C}$-vector space of polynomials $p\colon \mathbb{C}^{m+n} \to \mathbb{C}$ of degree $\leq d$ for $d \geq 1$. Let $\mathcal{L}^2_{\mathcal{P}}(K,\mu)$ denote the closed subspace of the Hilbert space $\mathcal{L}^2(K,\mu)$ generated by the restriction to K of polynomials $p \in \mathcal{P}_d(\mathbb{C}^{m+n})$ of degree$(p) \leq d$, with $d \geq 1$. Then every function $f \in \mathcal{L}^2_{\mathcal{P}}(K,\mu)$ has a power series expansion in the form

$$f = \sum_{k\geq 1} f_k A_k, \tag{3.8}$$

with

$$f_k = \frac{1}{\Delta_k(K)^2} \int_K f.\overline{A}_k d\mu; \tag{3.9}$$

here . is the dot product of vectors.

We are now in a position to state the main theorems of this section largely due to Zeriahi in his paper [Ze.1].

THEOREM 3.1: *If f: $X \to \mathbb{C}$ is a transcendental entire function on X with a series expansion*

$$f = \sum_{k\geq 1} f_k A_k, \tag{3.10}$$

with respect to the orthogonal polynomial basis $\{A_k\}_{k\geq 1}$, where

$$f_k = \frac{1}{\Delta_k^2(K)} \int_K f.\overline{A}_k d\mu, s_k = \text{degree } (A_k), \quad k \geq 1 \tag{3.11}$$

and . is the dot product of vectors, then f is of finite order ρ iff

$$\rho_1 = \limsup_{k\to\infty} \frac{s_k \log s_k}{-\log\Big(|f_k| a_k(K)\Big)} < +\infty, \tag{3.12}$$

and $\rho = \rho_1$.

THEOREM 3.2: *Let f: $X \to \mathbb{C}$ be a transcendental entire function on X which has a series expansion of the form*

$$f = \sum_{k\geq 1} f_k A_k, \tag{3.13}$$

in terms of the orthogonal polynomial basis $\{A_k\}_{k\geq 1}$, where

$$f_k = \frac{1}{\Delta_k^2(K)} \int_K f.\overline{A}_k d\mu, s_k = \text{degree } (A_k),$$

and the . stands for the dot product of vectors. Then f with a finite order ρ $(0 < \rho < +\infty)$ has finite K-type σ $(0 < \sigma < +\infty)$ iff

$$e\rho\sigma_1 = \limsup_{k\to\infty} s_k \Big(|f_k|a_k(K)\Big)^{\rho/S_k} < +\infty, \tag{3.14}$$

and $\sigma_1 = \sigma$.

For the proofs of Theorems 3.1 and 3.2 we need the following crucial Lemma of Zeriahi extending the classical Cauchy Inequality.

LEMMA 3.3: *Let $f = \sum_{k\geq 1} f_k A_k$ be a holomorphic function on X. Then for every $\theta > 1$, there exist an integer N_θ and a constant $C_\theta > 0$ such that*

$$|f_k| r^{s_k} \Delta_k(K) \leq C_\theta \frac{(r+1)^{N_\theta}}{(r-1)^{2n-1}} |f_k|_{\overline{\Omega}_{r\theta}}, \tag{3.15}$$

for every $r > 1$, $k \geq 1$, where C_θ and N_θ are independent of r, k and f.

We shall give here independent proofs of the two theorems above. The ideas of the proof are implicit in Ronkin's proof, in [Ro.1], of a similar theorem due to Goldberg. We begin with the proof of Theorem 3.1

Proof: We shall first demonstrate that $\rho_1 \leq \rho$. To do this we use the inequality

$$|f_k|a_k(K) \leq \frac{\mathcal{M}_{K,f}(r)}{r^{s_k}}, \quad r > 0. \tag{3.16}$$

If $\rho_1 = 0$, clearly $\rho_1 \leq \rho$ so there is nothing to prove. Now assume $0 < \rho_1 < \infty$. If $\rho_1 < \infty$, define $\rho_\epsilon = \rho_1 - \epsilon$, for small $\epsilon > 0$, so that $\rho_1 > 0$. Let $\rho_\epsilon > 0$ be arbitrary if $\rho_1 = +\infty$. Then it easily follows that for infinitely many indices $k \geq 1$, we have that $\log(|f_k|a_k(K)) < 0$ and

$$s_k \log s_k \geq \rho_\epsilon \log \frac{1}{\Big(|f_k|a_k(K)\Big)}. \tag{3.17}$$

We immediately see that for these indices k we can deduce that

$$\log\Big(|f_k|a_k(K)\Big) \geq -\frac{s_k \log s_k}{\rho_\epsilon} \tag{3.18}$$

and

$$\log \mathcal{M}_{K,f}(r_{s_k}) \geq \log\Big(|f_k|a_k(K)\Big) + s_k \log r \geq s_k\Big(\log r - \frac{\log s_k}{\rho_\epsilon}\Big). \tag{3.19}$$

Set $r_{s_k} = \left(es_k\right)^{1/\rho_\epsilon}$. Then we have $\log \mathcal{M}_{K,f}(r_{s_k}) \geq s_k/\rho_\epsilon$ and

$$\frac{\log\log \mathcal{M}_{K,f}(r_{s_k})}{\log r_{s_k}} \geq \rho_\epsilon\left(\frac{\log s_k - \log \rho_\epsilon}{\log s_k + 1}\right). \tag{3.20}$$

We can now conclude that

$$\rho = \limsup_{r\to\infty} \frac{\log\log \mathcal{M}_{K,f}(r)}{\log r} \geq \limsup_{r\to\infty} \frac{\log\log \mathcal{M}_{K,f}(r_{s_k})}{\log r_{s_k}} \geq \rho_\epsilon. \tag{3.21}$$

Since ρ_ϵ is an arbitrary real number, smaller than ρ, it follows that $\rho \geq \rho_1$. In the next step we have to show that $\rho \leq \rho_1$. First observe that if $\rho_1 = +\infty$, there is nothing to prove. Assume then that $\rho_1 < \infty$ and let $\epsilon > 0$. For a sufficiently large integer k, we have

$$0 \leq \frac{s_k \log s_k}{-\log\left(|f_k|a_k(K)\right)} \leq \rho_1 + \epsilon. \tag{3.22}$$

That is to say,

$$|f_k|a_k(K) \leq \left(s_k\right)^{-s_k/(\rho_1+\epsilon)}. \tag{3.23}$$

This condition implies that f is a transcendental entire function. Since adding a polynomial will not change the order of a function, we can assume that (3.23) holds for every $k \geq 1$ and set $a_0(K) = 0$. Thus for $r \geq 1$,

$$\mathcal{M}_{K,f}(r) \leq \sum_{k\geq 1} |f_k|a_k(K)r^{s_k} \leq \sum_{k\geq 1}\left(s_k\right)^{-s_k/(\rho_1+\epsilon)} r^{s_k} = \sum_1 + \sum_2, \tag{3.24}$$

where $\sum_1 = \sum_{1\leq k\leq (2r)^{\rho_1+\epsilon}} s_k^{-s_k/(\rho_1+\epsilon)} r^{s_k}$ and $\sum_2 = \sum_{k\geq (2r)^{\rho_1+\epsilon}} s_k^{-s_k/(\rho_1+\epsilon)} r^{s_k}$. In $\sum_2$, we have $rk^{-1/(\rho_1+\epsilon)} \leq \frac{1}{2}$, so that $\sum_2 \leq 1$. On the other hand,

$$\begin{aligned} \sum_1 \leq r^{(2r)^{\rho_1+\epsilon}} \sum_{k\geq 1} s_k^{-\frac{s_k}{\rho+\epsilon}} &\leq K_1 \exp\left((2r)^{\rho_1+\epsilon}\log r\right) \\ &\leq K_2 \exp\left(r^{\rho_1+2\epsilon}\right), \end{aligned} \tag{3.25}$$

for some constants $K_1 > 0$ and $K_2 > 0$. Hence it follows that $\rho \leq \rho_1 + 2\epsilon$. Since $\epsilon > 0$ is arbitrary, the theorem follows. ∎

The proof of Theorem 3.2 now follows:

Proof: We let $\gamma_1 = e\rho\sigma_1$, and then set

$$\gamma_1 = \limsup_{k\to\infty} s_k\Big(|f_k|a_k(K)\Big)^{\rho/s_k}. \tag{3.26}$$

If $\gamma_1 < \infty$, the function f can have at most the order ρ, and if $\gamma_1 > 0$ then f must have at least the order ρ. That is, if we let $\epsilon > 0$ and if $\gamma_1 > 0$, then for sufficiently large k

$$s_k\Big(|f_k|a_k(K)\Big)^{\rho/s_k} \le \gamma_1 + \epsilon, \tag{3.27}$$

so that one immediately deduces that

$$\log\Big(s_k(|f_k|a_k(K))^{\rho/s_k}\Big) \le \log(\gamma_1 + \epsilon). \tag{3.28}$$

This then implies the following inequality:

$$\frac{s_k \log s_k}{-\log(|f_k|a_k(K))} \le \frac{\rho}{(1 - \log(\frac{\gamma_1+\epsilon}{s_k}))}. \tag{3.29}$$

Furthermore, it follows from Theorem 3.1 that the order of f is at most ρ. The other statement is demonstrated similarly.

Let us now assume that $0 < \gamma_1 < \infty$ and then show that in this situation $\sigma \le \gamma_1/e\rho = \sigma_1$. For this let us take any $\epsilon > 0$; then for k sufficiently large we get

$$|f_k|a_k(K) \le \Big(\frac{\gamma_1 + \epsilon}{s_k}\Big)^{s_k/\rho}. \tag{3.30}$$

Suppose that (3.30) holds for $k \ge 1$ and that $a_0(K) = 1$, since adding a polynomial to f will not change the type of the function. Then

$$|f(z)| \le \sum_{k\ge 1} |f_k||A_k|_{\overline{\Omega}_k} \le \sum_{k\ge 1} |f_k|a_k(K)r^{s_k} \le \sum_{k\ge 1}\Big(r^\rho\Big(\frac{\gamma_1+\epsilon}{s_k}\Big)\Big)^{s_k/\rho}, \tag{3.31}$$

using (3.30). Now consider the function

$$\phi(s) := \Big(r^\rho\Big(\frac{\gamma_1+\epsilon}{s}\Big)\Big)^{s/\rho}, \qquad \text{for } s > 0.$$

This function attains its maximum value at

$$s = \frac{\gamma_1 + \epsilon}{e} r^\rho$$

and this value is equal to $\exp\left(\frac{\rho+\epsilon}{e\rho}r^\rho\right)$. Because of this we have, for any constant $K_1 > 0$,

$$\sum_1 = \sum_{1\le k\le(\gamma_1+2\epsilon)r^\rho}\left(\frac{(\gamma+\epsilon)}{s_k}\right)^{s_k/\rho} \le (\gamma_1+2\epsilon)r^\rho \exp\left(\frac{(\gamma_1+\epsilon)}{e\rho}r^\rho\right)$$

$$\le K_1 \exp\left(\frac{(\gamma_1+2\epsilon)}{e\rho}r^\rho\right); \tag{3.32}$$

$$\sum_2 = \sum_{k>(\gamma_1+2\epsilon)r^\rho}\left(\frac{(\gamma_1+2\epsilon)}{s_k}r^\rho\right)^{s_k/\rho} \le \sum_{k\ge1}\left(\frac{\gamma_1+\epsilon}{\gamma_1+2\epsilon}\right)^{s_k/\rho} = K_2 < \infty. \tag{3.33}$$

Thus it follows that $\sigma \le \gamma_1/e\rho$. To show the opposite inequality, first note that if $0<\epsilon<\gamma_1$, then there are infinitely many k such that

$$|f_k|a_k(K) \ge \left(\frac{\gamma_1-\epsilon}{s_k}\right)^{s_k/\rho}. \tag{3.34}$$

Hence in the inequality

$$|f_k|a_k(K) \le r^{-s_k}\mathcal{M}_{K,f}(r), \tag{3.35}$$

we can take $r_{s_k} = es_k/(\gamma_1-\epsilon)$. We thus obtain

$$\begin{aligned}\mathcal{M}_{K,f}(r_{s_k}) &\ge |f_k|a_k(K)r^{s_k} \ge \left(r_k^\rho\frac{(\gamma_1-1)}{s_k}\right)^{s_k/\rho}\\ &= e^{s_k/\rho} = \exp\left(\frac{(\gamma_1-\epsilon)}{e\rho}r_{s_k}^\rho\right).\end{aligned} \tag{3.36}$$

Therefore, $\sigma \ge \gamma_1/e\rho$. ∎

This completes the proofs of the two theorems.

4. (p,q)-Order and (p,q)-K-type on varieties

This final section is devoted to the main applications of the concepts of (p,q)-order and (p,q)-K-type to the study of the growth properties of trancendental entire functions on the complete intersection algebraic variety X in $\mathbb{C}^{m+n}$ of codimension $m>1$. We take K throughout to be a nonpluripolar, compact and $\mathcal{L}$-regular subset of X. We introduce one further technical concept, that of the maximum term of a transcendental entire function, in addition to the theory already developed in the previous sections, in our study of the growth properties of these transcendental entire functions defined on X.

Let $f\colon X \to \mathbb{C}$ be a transcendental entire function on a complete intersection algebraic variety X of codimension $m > 1$ in $\mathbb{C}^{m+n}$, which has a series expansion in the form

$$f := \sum_{k\geq 1} f_k A_k,$$

with respect to the orthogonal polynomial basis $\{A_k\}_{k\geq 1}$, with degree$(A_k) = s_k$, where

$$f_k = \frac{1}{\Delta_k^2(K)} \int_K f.\overline{A}_k d\mu \tag{4.1}$$

and

$$|f_k||A_k|_{\overline{\Omega}_r} \leq |f_k| a_k(K) r^{s_k}. \tag{4.2}$$

Set

$$C_{s_k} := |f_k| a_k(K);$$

then define the maximum term of the transcendental entire function f to be

$$\Theta_{K,f}(r) := \sup_{k\geq 1}\{C_{s_k} r^{s_k}\}, \quad r > 1. \tag{4.3}$$

We then have the following lemma which has the same proof as in [E-M, K.1].

LEMMA 4.1: *If $f\colon X \to \mathbb{C}$ is a transcendental entire function on a complete intersection algebraic variety X of codimension $m > 1$ in $\mathbb{C}^{m+n}$ of (p,q)-order ρ $(b < \rho < \infty)$ and (p,q)-K-type σ with respect to the proximate order $\rho(r)$, then*

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} \Theta_{K,f}(r)}{\left(\log^{[q-1]} r\right)^{\rho(r)}} = \sigma_K \equiv \sigma, \tag{4.4}$$

where $\Theta_{K,f}$ is the maximum term of f.

Proof: See similar proof in [E-M, K1]. ∎

Next define the function $\Gamma(\alpha) \equiv \Gamma(\alpha; p, q)$ on $0 \leq \alpha \leq \infty$, for the integer pair (p,q) with $p \geq q \geq 1$, by

$$\Gamma(\alpha) := \begin{cases} \alpha, & \text{if } p > q, \\ 1+\alpha, & \text{if } p = q = 2, \\ \max\{1,\alpha\}, & \text{if } 3 \leq p = q < \infty, \\ \infty, & \text{if } p = q = \infty. \end{cases}$$

We can now state the main results of this section. The following two theorems give, respectively, sufficiently complete coefficient characterizations of the (p,q)-order and the (p,q)-K-type of a transcendental entire function defined on a complete intersection algebraic variety X in $\mathbb{C}^{m+n}$. We note here that entire functions of index-pair $(1,1)$ are the polynomials.

THEOREM 4.2: *Let $f\colon X \to \mathbb{C}$ be a transcendental entire function on a complete intersection algebraic variety X of codimension $m > 1$ in $\mathbb{C}^{m+n}$. Let K be a compact, nonpluripolar and $\mathcal{L}$-regular subset of X. Suppose, further, that f has a series expansion*

$$f = \sum_{k \geq 1} f_k A_k, \tag{4.5}$$

in terms of the orthogonal polynomial basis $\{A_k\}_{k\geq 1}$, where

$$f_k = \frac{1}{\Delta_k^2(K)} \int_K f.\overline{A}_k d\mu, \tag{4.6}$$

and s_k = degree (A_k), *$k \geq 1$. If for a pair of integers (p,q) with $p \geq q \geq 1$, $\rho \equiv \rho(p,q)$ is defined by*

$$\rho \equiv \rho(p,q) := \limsup_{r\to\infty} \frac{\log^{[p]} \mathcal{M}_{K,f}(r)}{\log^{[q]} r}, \tag{4.7}$$

then

$$\rho \equiv \rho(p,q) = \Gamma(\eta(p,q)), \tag{4.8}$$

where

$$\eta \equiv \eta(p,q) := \limsup_{k\to\infty} \frac{\log^{[p]} s_k}{\log^{[q-1]}\left(\log\left(|f_k| a_k(K)\right)^{-1/s_k}\right)}. \tag{4.9}$$

Proof: The proof is an adaptation from the one-variable case along the same lines as the proof in [J-K-B.1], pages 61–62. However, in this case, at those places in the proof where the Cauchy estimate is used, we apply the Zeriahi version of the Cauchy estimate in Lemma 3.3. Now let us first assume that $\rho(p,q) < \infty$. Then for any given $\epsilon > 0$ and a fixed large $r_0 = r_0(\epsilon)$, we have from the definition of (p,q)-order that

$$\mathcal{M}_{K,f}(r) < \exp^{[p-2]}\left(\log^{[q-1]} r\right)^{\rho(p,q)+\epsilon}, \tag{4.10}$$

for all $r > r_0$. We can now apply Zeriahi's version of the Cauchy estimate to obtain

$$\log|f_k|a_k(K) < \exp^{[p-2]}\Big(\log^{[q-1]} r\Big)^{\rho(p,q)+\epsilon} - s_k \log r, \tag{4.11}$$

for all $r > r_0$ and for all $k > k_0 = k_0(\epsilon)$ for large fixed k_0.

Take for $(p,q) \neq (2,2)$,

$$r = \exp^{[q-1]}\Big(\log^{[p-2]}\Big(\frac{s_k}{\rho(p,q)+\epsilon}\Big)\Big)^{\frac{1}{\rho(p,q)+\epsilon}}. \tag{4.12}$$

Then, it follows from (4.11) that, for all k and $(p,q) \neq (2,2)$,

$$\log|f_k|a_k(K) < \frac{s_k}{\rho(p,q)+\epsilon} - s_k \exp^{[q-2]}\Big(\log^{[p-2]}\Big(\frac{s_k}{\rho(p,q)+\epsilon}\Big)\Big)^{\frac{1}{\rho(p,q)+\epsilon}}.$$

Because $\rho(p,q) \geq 1$, for $p = q$, the above inequality for $3 \leq p = q < \infty$ gives

$$\rho(p,q) \geq \max\{1, \eta(p,q)\}, \tag{4.13}$$

and for $p > q$ we obtain

$$\rho(p,q) \geq \eta(p,q). \tag{4.14}$$

Now consider the case $(p,q) = (2,2)$, and set

$$r = \exp\Big(\frac{s_k}{\rho(2,2)+\epsilon}\Big)^{\frac{1}{\rho(2,2)+\epsilon-1}}.$$

Then from (4.11), we deduce that, for all $k > k_0$,

$$\log\Big(|f_k|a_k(K)\Big)^{-1/s_k} > \Big(\frac{s_k}{\rho(2,2)+\epsilon}\Big)^{1/\rho(2,2)+\epsilon-1}\Big(\frac{\rho(2,2)+\epsilon-1}{\rho(2,2)+\epsilon}\Big).$$

From this inequality we have directly that

$$\rho(2,2) \geq 1 + \eta(2,2). \tag{4.15}$$

We combine the inequalities (4.13) , (4.14) and (4.15) to get

$$\rho(p,q) \geq \Gamma(\eta(p,q)). \tag{4.16}$$

It is then easy to see that (4.16) is valid for $\rho(p,q) = \infty$.

Next assume that $\eta(p,q) < \infty$. Then given any $\epsilon > 0$, the definition of $\eta(p,q)$ implies that

$$|f_k|a_k(K) < \exp\{-s_k \exp^{[q-2]}(\log^{[p-2]} s_k)^{1/\eta(p,q)+\epsilon}\}, \tag{4.17}$$

for all $k > k_0$. Since

$$\begin{aligned} \mathcal{M}_{K,f}(r) &\le \sum_{k=0}^{\infty} |f_k|a_k(K)r^{s_k} \\ &= \sum_{k=k_0+1}^{\beta} |f_k|a_k(K)r^{s_k} + \sum_{k=\beta+1}^{\infty} |f_k|a_k(K)r^{s_k} + P(k_0), \end{aligned}$$

where $P(k_0)$ is a polynomial of degree at most s_{k_0} and with β suitably chosen so that

$$s_\beta < \exp^{[p-2]}\left(\log^{[q-1]} 2r\right)^{\eta(p,q)+\epsilon} < s_{\beta+1},$$

we can now apply (4.17) to get an estimate for $\mathcal{M}_{K,f}(r)$ of the form

$$\begin{aligned} &\mathcal{M}_{K,f}(r) < \exp\{\exp^{[p-2]}(\log^{[q-1]} 2r)^{\eta(p,q)+\epsilon}\} \\ &\times \sum_{k=0}^{\infty} \exp\{-s_k \exp^{[q-2]}(\log^{[p-2]} s_k)^{1/\eta(p,q)+\epsilon}\} + \sum_{k}^{\infty} 2^{-k} + P(k_0). \end{aligned}$$

But now both series in the above estimate converge so that for sufficiently large r we can deduce the inequality

$$\log^{[2]} \mathcal{M}_{K,f}(r) < \exp^{[p-3]}\left(\log^{[q-1]} 2r\right)^{\eta(p,q)+\epsilon} + \log^{[2]} r + o(1).$$

This implies immediately that

$$\rho(p,q) \le \Gamma(\eta(p,q)). \tag{4.18}$$

This inequality is clearly true for $\eta(p,q) = \infty$. From the inequalities (4.16) and (4.18) we obtain the required result

$$\rho(p,q) = \Gamma(\eta(p,q)).$$

This completes the proof. ∎

THEOREM 4.3: *Let $f\colon X \to \mathbb{C}$ be a transcendental entire function on the complete intersection algebraic variety X of codimension $m > 1$ in $\mathbb{C}^{m+n}$. Let K be a compact, nonpluripolar and $\mathcal{L}$-regular subset of X. Furthermore, suppose that f has a series expansion*

$$f = \sum_{k\geq 1} f_k A_k, \tag{4.19}$$

in terms of the orthogonal polynomial basis $\{A_k\}_{k\geq 1}$, where

$$f_k = \frac{1}{\Delta_k^2(K)} \int_K f.\overline{A}_k d\mu, \tag{4.20}$$

with $s_k =$ degree $(A_k), k \geq 1$. Then f is of (p,q)-order ρ $(b < \rho < \infty)$ and (p,q)-K-type σ with respect to any proximate order $\rho(r)$ if and only if

$$\frac{\sigma}{\Upsilon} = \limsup_{k\to\infty} \left[\frac{\psi\left(\log^{[p-2]} s_k\right)}{\log^{[q-1]} C_{s_k}^{-1/s_k}} \right]^{\rho-A}, \tag{4.21}$$

where

$$\Upsilon = \begin{cases} \dfrac{(\rho-1)^{\rho-1}}{\rho^\rho}, & \text{if } (p,q) = (2,2), \\ \dfrac{1}{e\rho}, & \text{if } (p,q) = (2,1), \\ 1, & \text{otherwise}. \end{cases}$$

Here ψ is the function defined in (2.6).

Proof: The proof of this theorem proceeds along the same arguments presented in the proof of similar theorem in [E-M, K.1] in the case of transcendental entire functions studied in that paper. We will not reproduce the proof of that paper here, except to mention that the most important changes occur at the points where the classical Cauchy inequality is used. For the case here we use instead Zeriahi's extension of the Cauchy inequality in Lemma 3.3. ∎

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